How many proofs of the Nebitt's Inequality?

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1. Introduction

On March, 1903, Nesbitt proposed the following problem

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2} \qquad (1)$$

The above inequality is a famous problem. There are many people interested in and solved (1). In this paper, I would send readers some proofs of (1). And now, we begin.

2. Some proofs

Proof 1. Using the well - known inequality

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9 \text{ for } x,y,z>0.$$

We get

$$\left(\frac{a}{b+c}+1\right)+\left(\frac{b}{b+c}+1\right)+\left(\frac{c}{a+b}+1\right)=$$
$$=\frac{1}{2}\left[(a+b)+(b+c)+(c+a)\right]\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{c}{c+a}\right)\geq\frac{9}{2}.$$

Therefore, $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{9}{2}-3=\frac{3}{2}$.

Proof 2. Setting $x=b+c, y=c+a, z=a+b$, then

$$a=\frac{y+z-x}{2}, b=\frac{z+x-y}{2}, c=\frac{x+y-z}{2}.$$

(1) can be written as

$$\frac{y+z-x}{2x}+\frac{z+x-y}{2y}+\frac{x+y-z}{2z}\geq\frac{3}{2}$$
$$\Longleftrightarrow\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)+\left(\frac{z}{x}+\frac{x}{z}\right)\geq 6.$$

This inequality is clearly true by the well - known inequality

$$\frac{p}{q}+\frac{q}{p}\geq 2 \text{ for } p,q>0.$$

Proof 3. (1) is equivalent to

$$2\left[a(a+b)(a+c)+b(b+a)(b+c)+c(c+a)(c+b)\right]\geq$$
$$\geq 3(a+b)(b+c)(c+a)$$
$$\Longleftrightarrow 2\left(a^3+b^3+c^3\right)\geq ab(a+b)+bc(b+c)+ca(a+b).$$

The inequality follows by the well - known inequality

$$x^3+y^3\geq xy(x+y)\left(\Longleftrightarrow (x+y)(x-y)^2\geq 0\right) \text{ for } x,y\geq 0.$$

Proof 4. Using the **AM - GM** inequality, we obtain

$$\frac{a^2}{b+c}+\frac{b+c}{4}\geq 2\sqrt{\frac{a^2}{b+c}.\frac{b+c}{4}}=a.$$

Similarly, we get $\frac{b^2}{c+a}+\frac{c+a}{4}\geq b$, and $\frac{c^2}{a+b}+\frac{a+b}{4}\geq c$.

Adding three above inequalities, we have

$$\frac{a^2}{b+c}+\frac{b^2}{b+c}+\frac{c^2}{a+b}+\frac{a+b+c}{2}\geq a+b+c$$

$$\Longleftrightarrow \left(\frac{a^2}{b+c}+a\right)+\left(\frac{b^2}{b+c}+b\right)+\left(\frac{c^2}{a+b}+c\right)\geq \frac{3}{2}(a+b+c)$$
$$\Longleftrightarrow \frac{a(a+b+c)}{b+c}+\frac{b(a+b+c)}{b+c}+\frac{c(a+b+c)}{a+b}\geq \frac{3}{2}(a+b+c)$$
$$\Longleftrightarrow \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq \frac{3}{2}.$$

Proof 5. Setting $A=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}, B=\frac{b}{b+c}+\frac{c}{c+a}+\frac{a}{a+b}, C=\frac{c}{b+c}+\frac{a}{c+a}+\frac{b}{a+b}$ then $B+C=3, A+B=\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}$, and $A+C=\frac{a+c}{b+c}+\frac{b+a}{c+a}+\frac{c+b}{a+b}$.

By the **AM - GM** inequality, we have

$$A+B\geq 3\sqrt[3]{\frac{a+b}{b+c}.\frac{b+c}{c+a}.\frac{c+a}{a+b}}=3\ ,\ A+C\geq 3\sqrt[3]{\frac{a+c}{b+c}.\frac{b+a}{c+a}.\frac{c+b}{a+b}}=3.$$

Thus, $2A+B+C\geq 6$ or $A\geq \frac{3}{2}$.

Proof 6. By the **AM - GM** inequality, we have

$$\sqrt{a^3}+\sqrt{b^3}+\sqrt{b^3}\geq 3\sqrt[3]{\sqrt{a^3b^6}}=3b\sqrt{a}\ ,$$

and $\sqrt{a^3}+\sqrt{c^3}+\sqrt{c^3}\geq 3\sqrt[3]{\sqrt{a^3c^6}}=3c\sqrt{a}$.

Adding two inequalities, we get

$$2\left(\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}\right)\geq 3\sqrt{a}(b+c)$$

Thus, $\frac{a}{b+c}\geq \frac{3}{2}.\frac{\sqrt{a^3}}{\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}}$.

Similarly, we have $\frac{b}{c+a}\geq \frac{3}{2}.\frac{\sqrt{b^3}}{\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}}, \frac{c}{a+b}\geq \frac{3}{2}.\frac{\sqrt{c^3}}{\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}}$.

Adding three above inequalities, we obtain

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq \frac{3}{2}.\frac{\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}}{\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}}=\frac{3}{2}.$$

Proof 7. Using the well - known inequality

$$(x+y)^2\geq 4xy, \text{ for } x,y\geq 0.$$

We have

$$[2a+(b+c)]^2\geq 4.2a(b+c)\ \text{ or } 4a^2+4a(b+c)+(b+c)^2\geq 8a(b+c).$$

Hence, $4a(a+b+c)\geq (b+c)[8a-(b+c)]\Longrightarrow \frac{a}{b+c}\geq \frac{1}{4}.\frac{8a-b-c}{a+b+c}$.

Similarly, we get $\frac{b}{c+a}\geq \frac{1}{4}.\frac{8b-c-a}{a+b+c}, \frac{c}{a+b}\geq \frac{1}{4}.\frac{8c-a-b}{a+b+c}$.

Adding three inequalities, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq \frac{1}{4}.\frac{6(a+b+c)}{a+b+c}=\frac{3}{2}.$$

Proof 8. (By Cao Minh Quang) Using the **AM - GM** inequality, we have

$$\frac{2(a+b+c)}{3}=\frac{2a+(b+c)+(b+c)}{3}\geq \sqrt[3]{2a(b+c)^2}.$$

Thus, $\frac{a}{b+c}\geq \frac{3\sqrt{3}}{2}.\frac{a\sqrt{a}}{\sqrt{(a+b+c)^3}}$. Similarly, we get

$$\frac{b}{c+a} \geq \frac{3\sqrt{3}}{2}.\frac{b\sqrt{b}}{\sqrt{(a+b+c)^3}} \text{ and } \frac{c}{a+b} \geq \frac{3\sqrt{3}}{2}.\frac{c\sqrt{c}}{\sqrt{(a+b+c)^3}}.$$

Adding three inequalities, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3\sqrt{3}}{2}.\frac{a\sqrt{a}+b\sqrt{b}+c\sqrt{c}}{\sqrt{(a+b+c)^3}}.$$

We have to show that

$$\frac{3\sqrt{3}}{2}.\frac{a\sqrt{a}+b\sqrt{b}+c\sqrt{c}}{\sqrt{(a+b+c)^3}} \geq \frac{3}{2}$$

$$\Longleftrightarrow 3\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2 \geq (a+b+c)^3. \qquad (2)$$

We set $x=\sqrt{a}, y=\sqrt{b}, z=\sqrt{c}$. (2) can be written as

$$3\left(x^3+y^3+z^3\right)^2 \geq \left(x^2+y^2+z^2\right)^3.$$

By the **Cauchy - Schwarz** inequality, we have

$$\left(x^2+y^2+z^2\right)^2 \leq \left(x^3+y^3+z^3\right)(x+y+z) \qquad (3)$$

By the **Chebyshev's** inequality, we have

$$\left(x^2+y^2+z^2\right)(x+y+z) \leq 3\left(x^3+y^3+z^3\right) \qquad (4).$$

Multiplying (3) and (4) yields, we obtain

$$3\left(x^3+y^3+z^3\right)^2 \geq \left(x^2+y^2+z^2\right)^3.$$

Proof 9. (1) is equivalent to

$$\left(\frac{a}{b+c}-\frac{1}{2}\right)+\left(\frac{b}{c+a}-\frac{1}{2}\right)+\left(\frac{c}{a+b}-\frac{1}{2}\right) \geq 0$$

$$\Longleftrightarrow \frac{a-b+a-c}{b+c}+\frac{b-c+b-a}{b+c}+\frac{c-a+c-b}{a+b} \geq 0$$

$$\Longleftrightarrow (a-b)\left(\frac{1}{b+c}-\frac{1}{c+a}\right)+(b-c)\left(\frac{1}{a+c}-\frac{1}{c+b}\right)+$$
$$+(a-c)\left(\frac{1}{b+c}-\frac{1}{a+b}\right) \geq 0$$

$$\Longleftrightarrow \frac{(a-b)^2}{(b+c)(a+c)}+\frac{(b-c)^2}{(a+c)(a+b)}+\frac{(c-a)^2}{(b+c)(a+b)} \geq 0.$$

The last inequality is clearly true.

Proof 10. By the **Cauchy - Schwarz** inequality, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{(a+b+c)^2}{a(b+c)+b(c+a)+c(c+a)} =$$
$$= \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3(ab+bc+ca)}{2(ab+bc+ca)} = \frac{3}{2}.$$

Proof 11. Without loss of generality, we can assume that $a \geq b \geq c$, then $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$. By the **Chebyshev's** inequality and the **AM - GM** inequality, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{1}{3}(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right) =$$
$$= \frac{1}{6}\left[(a+b)+(b+c)+(c+a)\right]\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{c}{c+a}\right) \geq \frac{9}{6} = \frac{3}{2}.$$

Proof 12. Setting $x = \dfrac{a}{b+c}, y = \dfrac{b}{c+a}, z = \dfrac{c}{a+b}$, and $A = \dfrac{x+y+z}{3}$. We need to prove that $A \geq \dfrac{1}{2}$. We have

$$\frac{x}{1+x} + \frac{y}{1+y} + \frac{z}{1+z} = \frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} = 1.$$

Thus, $1 = 2xyz + xy + yz + zx$.

By the **AM - GM** inequality, we get

$$1 = 2xyz + xy + yz + zx \leq 2A^3 + 3A^3 \Longrightarrow (2A-1)(A+1)^2 \geq 0$$

Since $A > 0$, hence $A \geq \dfrac{1}{2}$.

Proof 13. Using the same substitution as the 12^{th} proof, and setting $f(t) = \dfrac{t}{1+t}$. It is easy to show that $f(t)$ is increase and concave on $(0, +\infty)$. By the **Jensen's** inequality, we have

$$f\left(\frac{1}{2}\right) = \frac{1}{3} = \frac{1}{3}\left[f(x) + f(y) + f(z)\right] \leq f\left(\frac{x+y+z}{3}\right)$$

$$\Longrightarrow \frac{1}{2} \leq \frac{x+y+z}{3} \text{ or } \frac{3}{2} \leq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

Proof 14. (By Cao Minh Quang) Firstly, we state and prove a lemma.

Lemma. If $x_i, y_i, (i = 1, 2, 3)$ are positive real numbers satisfy that $x_1 \geq x_2 \geq x_3, y_1 \geq y_2 \geq y_3$, then

$$x_1y_1 + x_2y_2 + x_3y_3 \geq x_1y_{i_1} + x_2y_{i_2} + x_3y_{i_3}, \qquad (5)$$

where (i_1, i_2, i_3) is a permutation of $(1, 2, 3)$.

Proof (5). We set $z_1 = y_{i_1}, z_2 = y_{i_2}, z_3 = y_{i_3}$. (5) becomes

$$x_1y_1 + x_2y_2 + x_3y_3 \geq x_1z_1 + x_2z_2 + x_3z_3$$

$$\Longleftrightarrow x_1(y_1 - z_1) + x_2(y_2 - z_2) + x_3(y_3 - z_3) \geq 0.$$

It is easy to see that $y_1 \geq z_1, y_1 + y_2 \geq z_1 + z_2$ and $y_1 + y_2 + y_3 = z_1 + z_2 + z_3$. Therefore,

$$x_1(y_1 - z_1) + x_2(y_2 - z_2) + x_3(y_3 - z_3) \geq x_2(y_1 - z_1) + x_2(y_2 - z_2) + \\ + x_3(y_3 - z_3) = x_2\left[(y_1 + y_2) - (z_1 + z_2)\right] + x_3(y_3 - z_3) \geq x_3[(y_1 + y_2) - \\ - (z_1 + z_2)] + x_3(y_3 - z_3) = x_3\left[(y_1 + y_2 + y_3) - (z_1 + z_2 + z_3)\right] = 0.$$

Let us now prove (1). Without loss of generality, we can assume that $a \geq b \geq c$, then $\dfrac{1}{b+c} \geq \dfrac{1}{c+a} \geq \dfrac{1}{a+b}$. Using (5), we get

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b} \text{ and}$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{c}{b+c} + \frac{a}{c+a} + \frac{b}{a+b}.$$

Adding two inequalities, we obtain

$$2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq \frac{b+c}{b+c} + \frac{c+a}{c+a} + \frac{a+b}{a+b} = 3 \text{ or}$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 15. Without loss of generality, we can assume that $a \geq b \geq c$. We set $x = \dfrac{a}{c}, y = \dfrac{b}{c}$, then $x \geq y \geq 1$.

(1) becomes

$$\frac{x}{y+1}+\frac{y}{x+1}+\frac{1}{x+y}\geq\frac{3}{2} \qquad (6)$$

Using the **AM - GM** inequality, we have

$$\frac{x+1}{y+1}+\frac{y+1}{x+1}\geq 2\Longrightarrow\frac{x}{y+1}+\frac{y}{x+1}\geq 2-\frac{1}{x+1}-\frac{1}{y+1}.$$

It suffices to prove that

$$2-\frac{1}{x+1}-\frac{1}{y+1}\geq\frac{3}{2}-\frac{1}{x+y}$$

$$\Longleftrightarrow\frac{1}{2}-\frac{1}{y+1}\geq\frac{1}{x+1}-\frac{1}{x+y}$$

$$\Longleftrightarrow\frac{y-1}{2\left(y+1\right)}\geq\frac{y-1}{\left(x+1\right)\left(x+y\right)}$$

$$\Longleftrightarrow\frac{\left(y-1\right)\left[\left(x+1\right)\left(x+y\right)-2\left(y+1\right)\right]}{2\left(x+y\right)\left(x+1\right)\left(y+1\right)}\geq 0.$$

The last inequality i true since $x\geq y\geq 1$.

To prove (6), beside the above proof, we also have an another proof.

Proof 16. We set $m=x+y, n=xy$.

(6) becomes

$$\frac{m^2-2n+m}{m+n+1}+\frac{1}{m}\geq\frac{3}{2}$$

$$\Longleftrightarrow 2m^3-m^2-m+2\geq n\left(7m-2\right)$$

We note that $7m-2>0$ and $m^2\geq 4n$. It suffices to prove that

$$4\left(2m^3-m^2-m+2\right)\geq m^2\left(7m-2\right)\Longleftrightarrow\left(m-2\right)^2\left(m+2\right)\geq 0.$$

This inequality is clearly true.

Proof 17. (By Cao Minh Quang)

By setting $x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a}$, then $xyz=1$.

(1) can be written as

$$\frac{x}{xz+1}+\frac{y}{yx+1}+\frac{z}{zy+1}\geq\frac{3}{2}$$

$$\Longleftrightarrow 2\left(x^2y+y^2z+z^2x\right)\geq\left(x+y+z\right)+\left(xy+yz+zx\right) \qquad (7)$$

Using the **AM - GM** inequality, we have

$$\left(x^2y+y^2z+z^2x\right)=\frac{1}{3}[\left(x^2y+y^2z+y^2z\right)+\left(y^2z+z^2x+z^2x\right)+$$
$$+\left(z^2x+x^2y+x^2y\right)]\geq\sqrt[3]{x^2y^5z^2}+\sqrt[3]{y^2z^5x^2}+\sqrt[3]{z^2x^5y^2}=x+y+z.$$

Using the **AM - GM** inequality again, we have

$$\left(x^2y+y^2z+z^2x\right)=\frac{1}{3}[\left(x^2y+z^2x+z^2x\right)+\left(y^2z+x^2y+x^2y\right)+$$
$$+\left(z^2x+y^2z+y^2z\right)]\geq\sqrt[3]{x^4z^4y}+\sqrt[3]{x^4y^4z}+\sqrt[3]{xy^4z^4}=xz+xy+yz.$$

Adding two inequalities, we get (7).

Since (1) in the homogeneous, we can assume that $a+b+c=1$.

We have to prove

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}, \text{ where } a+b+c=1 \qquad (1')$$

We have some proofs of (1')

Proof 18. For $0<x<1$, we note that

$$4x-\left(1-x\right)\left(9x-1\right)=\left(3x-1\right)^2.$$

Hence, $4x \geq (1-x)(9x-1)$ or $\dfrac{x}{1-x} \geq \dfrac{9x-1}{4}$.
Therefore,
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c} \geq$$
$$\geq \frac{9a-1}{4}+\frac{9b-1}{4}+\frac{9c-1}{4}=\frac{9(a+b+c)-3}{4}=\frac{3}{2}.$$
Proof 19. (By Cao Minh Quang) For $0<x<1$, we note that
$$(3x-1)^2 \geq 0 \Longleftrightarrow 3x+1 \geq 9x(1-x).$$
Hence $\dfrac{1}{1-x} \geq \dfrac{9x}{3x+1}$ or $\dfrac{x}{1-x} \geq \dfrac{9x^2}{3x+1}$.
Therefore, by **Cauchy - Schwarz** Inequality, we have
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c} \geq$$
$$\geq \frac{9a^2}{3a+1}+\frac{9b^2}{3b+1}+\frac{9c^2}{3c+1} \geq \frac{(3a+3b+3c)^2}{3(a+b+c)+3}=\frac{3}{2}.$$
Proof 20. (By Cao Minh Quang) For $0<x<1$, by the **AM - GM** inequality, we get
$$(2-2x)(1+x)(1+x) \leq \left[\frac{(2-2x)+(1+x)+(1+x)}{3}\right]^3=\frac{64}{27}.$$
Therefore, $\dfrac{x}{1-x} \geq \dfrac{27}{32}x(1+x)^2$.
Using the well - known inequality
$$\frac{x^r+y^r+z^r}{3} \geq \left(\frac{x+y+z}{3}\right)^r, \text{ for } x,y,z \geq 0,\ r \geq 1.$$
We get
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c} \geq$$
$$\geq \frac{27}{32}\sum_{a,b,c} a(1+a)^2=\frac{27}{32}\left[\sum_{a,b,c} a^3+2\sum_{a,b,c} a^2+\sum_{a,b,c} a\right] \geq$$
$$\geq \frac{27}{32}\left[3\left(\frac{1}{3}\right)^3+6\left(\frac{1}{3}\right)^2+1\right]=\frac{3}{2}.$$
Proof 21. (By Cao Minh Quang) By the **AM - GM** inequality, we get
$$\frac{a}{b+c}+\frac{9a(b+c)}{4} \geq 2\sqrt{\frac{a}{b+c}.\frac{9a(b+c)}{4}}=3a.$$
Similarly, we have $\dfrac{b}{c+a}+\dfrac{9b(c+a)}{4} \geq 3b$, $\dfrac{c}{a+b}+\dfrac{9c(a+b)}{4} \geq 3c$.
Adding three inequalities and noting that
$$(a+b+c)=(a+b+c)^2 \geq 3(ab+bc+ca)$$
We obtain
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq 3(a+b+c)-\frac{9}{2}(ab+bc+ca) \geq$$
$$=\frac{3}{2}(a+b+c)+\left[\frac{3}{2}(a+b+c)-\frac{9}{2}(ab+bc+ca)\right] \geq \frac{3}{2}.$$

Proof 22. Setting $f(t) = \dfrac{t}{1-t}$. It is easy to show that $f(t)$ is a convex function on $(0,1)$. By the **Jensen's** inequality, we obtain

$$f(a) + f(b) + f(c) \geq 3f\left(\frac{a+b+c}{3}\right)$$

or $\dfrac{a}{1-a} + \dfrac{b}{1-b} + \dfrac{c}{1-c} \geq \dfrac{3}{2}$.

Proof 23. Without loss of generality, we can assume that $a \geq b \geq c$. We get $a \geq \dfrac{1}{3}, c \leq \dfrac{1}{3}$, which follows that $(a,b,c) \succeq \left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$.

Since $f(t) = \dfrac{t}{1-t}$ is a convex function on $(0,1)$, by the **Karamata's** inequality, we obtain

$$f(a) + f(b) + f(c) \geq f\left(\frac{1}{3}\right) + f\left(\frac{1}{3}\right) + f\left(\frac{1}{3}\right)$$

or $\dfrac{a}{1-a} + \dfrac{b}{1-b} + \dfrac{c}{1-c} \geq \dfrac{3}{2}$.

We are done.

Proof 24. (By Cao Minh Quang) To prove (1'), we need to prove a lemma.

Lemma. (Cao Minh Quang and Tran Tuan Anh)

If a, b, c are positive real numbers such that $a + b + c = 1$, then

$$\frac{a+bc}{b+c} + \frac{b+ca}{c+a} + \frac{c+ab}{a+b} \geq 2.$$

Proof. Using the well - known inequality

$$x^2 + y^2 + z^2 \geq xy + yz + zx, \text{ for } x, y, z \geq 0$$

We get

$$\frac{a+bc}{b+c} + \frac{b+ca}{c+a} + \frac{c+ab}{a+b} = \frac{(a+b)(a+c)}{b+c} + \frac{(b+c)(b+a)}{c+a} +$$

$$+ \frac{(c+a)(c+b)}{a+b} \geq (a+b) + (b+c) + (c+a) = 2.$$

We now prove (1'). Using the well - known inequality

$$\frac{xy}{x+y} \leq \frac{1}{4}(x+y), \text{ for } x, y > 0$$

We obtain

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 2 - \left(\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b}\right) \geq$$

$$\geq 2 - \frac{1}{4}\left[(b+c) + (c+a) + (a+b)\right] = \frac{3}{2}.$$

Last, we use "Mixing Variables Theorem" to prove (1).

Proof 25. By setting $E(a,b,c) = \dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}$, $t = \dfrac{a+b}{2}$, and $v = \dfrac{a+b+c}{3}$. It is easy to see that

$$E(a,b,c) = \frac{a^2+b^2+c(a+b)}{ab+c^2+c(a+b)} + \frac{c}{a+b} \geq \frac{2t^2+2tc}{t^2+c^2+2ct} + \frac{c}{2t} = E(t,t,c).$$

Thus, $E(a,b,c) \geq E(v,v,v) = \dfrac{3}{2}$.

3. Some general results of the Nesbitt's Inequality

In recent years, by some powerful tools to prove inequalities, people found out some inequalities which is "stronger" than (1). There are

Problem 1. [Titu Vareescu, Mircea Lascu] Let a, b, c, α be positive real numbers such that $abc = 1$ and $\alpha \geq 1$. Prove that

$$\frac{a^\alpha}{b+c} + \frac{b^\alpha}{c+a} + \frac{c^\alpha}{a+b} \geq \frac{3}{2}.$$

Problem 2. [Tran Tuan Anh] Let a, b, c, k be positive real numbers such that $k \geq \frac{2}{3}$. Prove that

$$\left(\frac{a}{b+c}\right)^k + \left(\frac{b}{c+a}\right)^k + \left(\frac{c}{a+b}\right)^k \geq \frac{3}{2^k}.$$

And a problem is "stronger" than problem 2 is

Problem 3. [Vasile Cirtoaje] Let a, b, c, r be positive real numbers such that $r \geq \frac{\ln 3}{\ln 2} - 1$. Prove that

$$\left(\frac{2a}{b+c}\right)^r + \left(\frac{2b}{c+a}\right)^r + \left(\frac{2c}{a+b}\right)^r \geq 3.$$

Problem 4. [Tran Nam Dung] Let a, b, c be positive real numbers. Prove that

$$\frac{1}{2} + \frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 2 - \frac{1}{2}.\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

But problem 4 is weaker than problem 5

Problem 5. [Vasile Cirtoaje] Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{13}{6} - \frac{2}{3}.\frac{ab+bc+ca}{a^2+b^2+c^2}.$$

Problem 6. [Cezar Lupu] Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{a^2+bc}{(a+b)(a+c)} + \frac{b^2+ca}{(b+a)(b+c)} + \frac{c^2+ab}{(c+a)(c+b)}.$$

Problem 7. [Cao Minh Quang] Let a, b, c be positive real numbers and m, n be nonnegative real numbers such that $a + b + c = 1, 6m \geq 5n$. Prove that

$$\frac{ma+nbc}{b+c} + \frac{mb+nca}{c+a} + \frac{mc+nab}{a+b} \geq \frac{3m+n}{2}.$$

The story of "the Nesbitt's Inequality" is still. Hoping anyone will find out an another of (1) on a recent day.

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