Oliforum contest -3rd edition

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November 16, 2012

Proposition 1. *Show that there exist infinitely many positive integers n such that n^2 divides $2^n + 3^n$*

Proposition 2. *Show that for every polynomial $f(x)$ with integer coefficients, there exists infinitely many positive integer n such that the sum of digits of $f(n)$ is constant.*

Proposition 3. *Show that if equiangular hexagon has sides a, b, c, d, e, f in order then $a - d = e - b = c - f$.*

Proposition 4. *Show that if $a \geq b \geq c \geq 0$ then*

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$$

Proposition 5. *Consider a cyclic quadrilateral $ABCD$ and define points $X = AB \cap CD$, $Y = AD \cap BC$, and suppose that there exists a circle with center Z inscribed in $ABCD$. Show that the Z belongs to the circle with diameter XY, which is orthogonal to circumcircle of $ABCD$.*

Proposition 6. *Suppose that every integer is colored using one of 4 colors. Let m, n be distinct odd integers such that $m + n \neq 0$. Prove that there exist integers a, b of the same color such that $a + b$ equals one of the numbers $m, n, m - n, m + n$.*

Proof. **Problem 1.**

$n_1 := 5$ works. Suppose $n_i \in \mathbb{N}_0$ works, then since $2^n + 3^n > n^2$ for all integers $n > 0$, there exists a prime $p_i > 2$ such that $p_i n_i^2 \mid 2^{n_i} + 3^{n_i}$. Now

$$2^{p_i n_i} + 3^{p_i n_i} = (2^{n_i} + 3^{n_i}) \left(\sum_{0 \le j \le p-1} (-1)^j 2^{n_i j} 3^{(p-1-j)n_i} \right)$$

□

Then $p_i \mid 2^{n_i} + 3^{n_i}$ and

$$\sum_{0 \le j \le p-1} (-1)^j 2^{n_i j} 3^{(p-1-j)n_i} \equiv \sum_{0 \le j \le p-1} (-1)^j 2^{n_i j} (-2)^{n_i (p-1-j)}$$

$$\equiv \sum_{0 \le j \le p-1} 2^{n_i (p-1)} \equiv 0 \pmod{p_i}$$

It implies that $n_{i+1} := n_i p_i$ is a suitable integer.

Proof. **Problem 2.**

Lemma: there exists a integer y such that $q_y(x) := p(x+y) \in \mathbb{N}[x]$.

Proof: Define $p(x) = \sum_{0 \le j \le d} a_j x^j$ for some integers $a_0, a_1, \ldots, a_d$ such that wlog $d > 0$.

If $a_i \ge 0$ for all $i = 0, 1, \ldots, d$, then it's enough to choose $y = 0$. Otherwise we can define $r := \max\{n \in \mathbb{N} \cap [0, d-1] : a_n < 0$. Notice that the polynomial

$$q_{-r}(x) = p(x - r) = \sum_{j=0}^{r} a_j (x - a_r)^j + \sum_{j=r+1}^{d} a_j (x - a_r)^j$$

has the coefficient of x^r equals to

$$a_r + \left(\sum_{j=r+1}^{d} a_j \binom{j}{r} (-a_r)^{j-r} \right) \ge a_r + a_d \cdot (-a_r) \ge 0$$

and obviously the coefficients of x^j are non negative for all $r+1 \le j \le d$.

It means that the greatest negative index of $q_{-r}(x)$ (if it exists) is not greater than $r - 1$.

Iterating this process on $q_{-r}(x)$ and so on, this alghoritm must end in a finite number of steps, and we get the result. []

Turning back to the original problem, once we know that $p(x) = q_y(x-y)$ and $q_y(x) \in \mathbb{N}[x]$, it's enough to choose $x = 10^m + y$ for all m sufficiently large: in that case indeed $s(p(x))$ equals to the sum of coefficients of $q_y(x)$.

□

Proof. **Problem 3.** Expand every second side of the hexagon to obtain a equilateral triangle ABC. Since AB=BC=CA, the result follows. □

Proof. **Problem 4.** We have

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) =$$
$$= \big(a^2b(a-b) - ab^2(a-b)\big) + \big(b^2c(b-c - ab^2(b-c)\big) + \big(c^a(c-a) - ab^2(c-a)\big) =$$
$$= ab(a-b)^2 + (ab+ac-b^2)(a-c)(b-c) \geq 0$$

□

Proof. **Problem 5.**

$$\angle AXY + \angle AYX = \pi - \angle XAY = \angle BCD$$

$$\angle CXY + \angle CYX = \pi - \angle XCY = \pi - \angle BCD$$

and recalling that ZY and ZX are bisectors of $\angle AXC$ and $\angle AYC$, then

$$\angle ZXY + \angle ZYX = \pi/2$$

and that's enough to conclude that Z belongs to the circle with diameter XY.

Moreover, define $W = AC \cap BD$. Then W is the polar of XY with respect to the external circle of $ABCD$, and similarly Y of XW and X of WY. Define O the circumcenter of $ABCD$. We have that $OX \perp WY$ (since it's polar); define also $T = OX \cap WY$. Then T is the inverse of X with respect to the circle, but we have that $\angle XTY = \pi/2$, so T belongs to the circle with diameter XY. The same holds for Y. It's enough to conclude that the circle with diameter XY is fixed for inversion in the circle external to $ABCD$, i.e. they are orthogonal.

□

Proof. **Problem 6.** Suppose for the sake of contradiction that $a - b \in \{m, n, m-n, m+n\}$ implies that a and b are colored differently. First, note that the condition implies that for any a, the numbers $a, a+n, a+m, a+n+m$ are all colored differently. Then, by a simple induction on k, the numbers $a+km+n$ and $a+km$ have the same set of colors as a and $a+n$ do for k even, and they have different colors if k is odd. Then, by a simple induction on j, if k is any odd integer, the numbers $a+km+jn$ and $a+jn$ have the same set of colors as a and $a+km$ do for j even, and they have the opposite colors if j is odd.

Then, putting $k = n$ and $j = m$, we have a contradiction: that $a+mn$ and $a+nm$ have different colors. $\square$