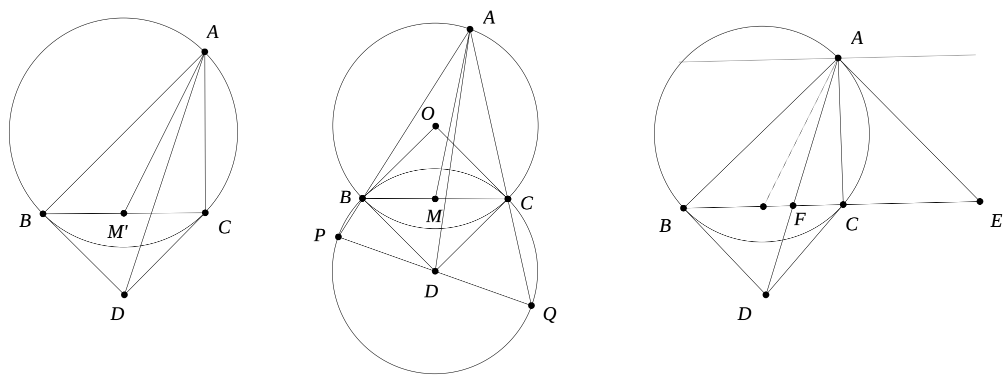
Lemmas in Euclidean Geometry[1]

Yufei Zhao

yufeiz@mit.edu

1. **Construction of the symmedian.**

Let ABC be a triangle and Γ its circumcircle. Let the tangent to Γ at B and C meet at D. Then AD coincides with a symmedian of $\triangle ABC$. (The *symmedian* is the reflection of the median across the angle bisector, all through the same vertex.)



We give three proofs. The first proof is a straightforward computation using Sine Law. The second proof uses similar triangles. The third proof uses projective geometry.

First proof. Let the reflection of AD across the angle bisector of $\angle BAC$ meet BC at M'. Then

$$\frac{BM'}{M'C} = \frac{AM'\frac{\sin\angle BAM'}{\sin\angle ABC}}{AM'\frac{\sin\angle CAM'}{\sin\angle ACB}} = \frac{\sin\angle BAM'}{\sin\angle ACD}\frac{\sin\angle ABD}{\sin\angle CAM'} = \frac{\sin\angle CAD}{\sin\angle ACD}\frac{\sin\angle ABD}{\sin\angle BAD} = \frac{CD}{AD}\frac{AD}{BD} = 1$$

Therefore, AM' is the median, and thus AD is the symmedian. $\square$

Second proof. Let O be the circumcenter of ABC and let ω be the circle centered at D with radius DB. Let lines AB and AC meet ω at P and Q, respectively. Since $\angle PBQ = \angle DQC + \angle BAC = \frac{1}{2}(\angle BDC + \angle DOC) = 90^\circ$, we see that PQ is a diameter of ω and hence passes through D. Since $\angle ABC = \angle AQP$ and $\angle ACB = \angle APQ$, we see that triangles ABC and AQP are similar. If M is the midpoint of BC, noting that D is the midpoint of QP, the similarity implies that $\angle BAM = \angle QAD$, from which the result follows. $\square$

Third proof. Let the tangent of Γ at A meet line BC at E. Then E is the pole of AD (since the polar of A is AE and the pole of D is BC). Let BC meet AD at F. Then point B, C, E, F are harmonic. This means that line AB, AC, AE, AF are harmonic. Consider the reflections of the four line across the angle bisector of $\angle BAC$. Their images must be harmonic too. It's easy to check that AE maps onto a line parallel to BC. Since BC must meet these four lines at harmonic points, it follows that the reflection of AF must pass through the midpoint of BC. Therefore, AF is a symmedian. $\square$

[1]Updated July 26, 2008

Related problems:

(i) (Poland 2000) Let ABC be a triangle with $AC = BC$, and P a point inside the triangle such that $\angle PAB = \angle PBC$. If M is the midpoint of AB, then show that $\angle APM + \angle BPC = 180°$.

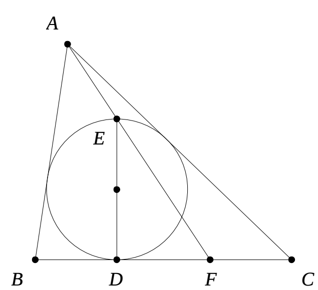
(ii) (IMO Shortlist 2003) Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose center does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

(iii) (Vietnam TST 2001) In the plane, two circles intersect at A and B, and a common tangent intersects the circles at P and Q. Let the tangents at P and Q to the circumcircle of triangle APQ intersect at S, and let H be the reflection of B across the line PQ. Prove that the points A, S, and H are collinear.

(iv) (USA TST 2007) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lies on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

(v) (USA 2008) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

2. **Diameter of the incircle.**



Let the incircle of triangle ABC touch side BC at D, and let DE be a diameter of the circle. If line AE meets BC at F, then $BD = CF$.

Proof. Consider the dilation with center A that carries the incircle to an excircle. The diameter DE of the incircle must be mapped to the diameter of the excircle that is perpendicular to BC. It follows that E must get mapped to the point of tangency between the excircle and BC. Since the image of E must lie on the line AE, it must be F. That is, the excircle is tangent to BC at F. Then, it follows easily that $BD = CF$. □

Related problems:

(i) (IMO Shortlist 2005) In a triangle ABC satisfying $AB + BC = 3AC$ the incircle has centre I and touches the sides AB and BC at D and E, respectively. Let K and L be the symmetric points of D and E with respect to I. Prove that the quadrilateral $ACKL$ is cyclic.

(ii) (IMO 1992) In the plane let $\mathcal{C}$ be a circle, ℓ a line tangent to the circle $\mathcal{C}$, and M a point on ℓ. Find the locus of all points P with the following property: there exists two points Q, R on ℓ such that M is the midpoint of QR and $\mathcal{C}$ is the inscribed circle of triangle PQR.

(iii) (USAMO 1999) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

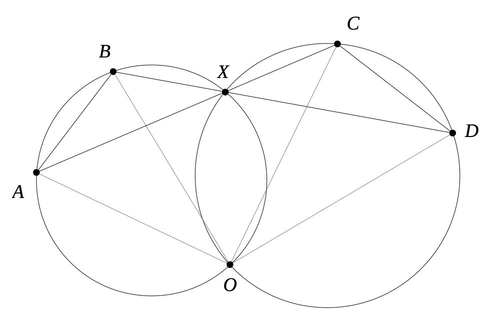
(iv) (USAMO 2001) Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

(v) (Tournament of Towns 2003 Fall) Triangle ABC has orthocenter H, incenter I and circumcenter O. Let K be the point where the incircle touches BC. If IO is parallel to BC, then prove that AO is parallel to HK.

(vi) (IMO 2008) Let $ABCD$ be a convex quadrilateral with $|BA| \neq |BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

3. **Dude, where's my spiral center?**

Let AB and CD be two segments, and let lines AC and BD meet at X. Let the circumcircles of ABX and CDX meet again at O. Then O is the center of the spiral similarity that carries AB to CD.



Proof. Since $ABOX$ and $CDXO$ are cyclic, we have $\angle OBD = \angle OAC$ and $\angle OCA = \angle ODB$. It follows that triangles AOC and BOD are similar. The result is immediate. □

Remember that spiral similarities always come in pairs: if there is a spiral similarity that carries AB to CD, then there is one that carries AC to BD.

Related problems:

(i) (IMO Shortlist 2006) Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle CBA = \angle DCA = \angle EDA.$$

Diagonals BD and CE meet at P. Prove that line AP bisects side CD.

(ii) (China 1992) Convex quadrilateral $ABCD$ is inscribed in circle ω with center O. Diagonals AC and BD meet at P. The circumcircles of triangles ABP and CDP meet at P and Q. Assume that points O, P, and Q are distinct. Prove that $\angle OQP = 90^\circ$.

(iii) Let $ABCD$ be a quadrilateral. Let diagonals AC and BD meet at P. Let O_1 and O_2 be the circumcenters of APD and BPC. Let M, N and O be the midpoints of AC, BD and O_1O_2. Show that O is the circumcenter of MPN.

(iv) (USAMO 2006) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

(v) (IMO 2005) Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

(vi) (IMO Shortlist 2002) Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

(vii) (USA TST 2006) In acute triangle ABC, segments AD, BE, and CF are its altitudes, and H is its orthocenter. Circle ω, centered at O, passes through A and H and intersects sides AB and AC again at Q and P (other than A), respectively. The circumcircle of triangle OPQ is tangent to segment BC at R. Prove that $CR/BR = ED/FD$.

(viii) (IMO Shortlist 2006) Points A_1, B_1 and C_1 are chosen on sides BC, CA, and AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, and CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, and C_2, respectively ($A_2 \neq A, B_2 \neq B$, and $C_2 \neq C$). Points A_3, B_3, and C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of sides BC, CA, and AB, respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

4. **Arc midpoints are equidistant to vertices and in/excenters**

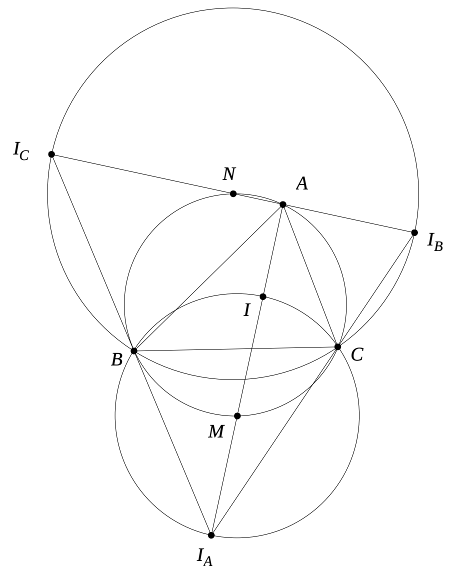
Let ABC be a triangle, I its incenter, and I_A, I_B, I_C its excenters. On the circumcircle of ABC, let M be the midpoint of the arc BC not containing A and let N be the midpoint of the arc BC containing A. Then $MB = MC = MI = MI_A$ and $NB = NC = NI_B = NI_C$.

Proof. Straightforward angle-chasing (do it yourself!). Another perspective is to consider the circumcircle of ABC as the nine-point-circle of $I_AI_BI_C$. □

Related problems:

(i) (APMO 2007) Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

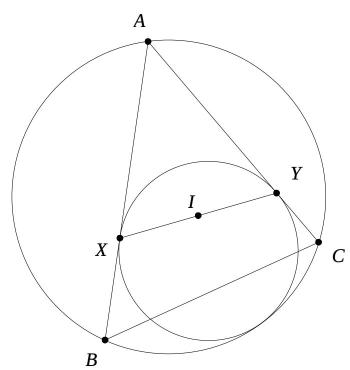
(ii) (IMO 2006) Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

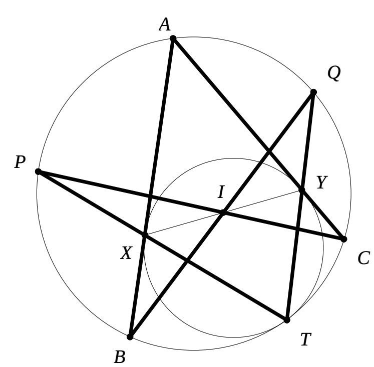


(iii) (Romanian TST 1996) Let $ABCD$ be a cyclic quadrilateral and let $\mathcal{M}$ be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (16 points in total). Prove that there are two sets $\mathcal{K}$ and $\mathcal{L}$ of four parallel lines each, such that every line in $\mathcal{K} \cup \mathcal{L}$ contains exactly four points of $\mathcal{M}$.

5. **I is the midpoint of the touch-chord of the mixtilinear incircles**

Let ABC be a triangle and I its incenter. Let Γ be the circle tangent to sides AB, AC, as well as the circumcircle of ABC. Let Γ touch AB and AC at X and Y, respectively. Then I is the midpoint of XY.





Proof. Let the point of tangency between the two circles be T. Extend TX and TY to meet the circumcircle of ABC again at P and Q respectively. Note that P and Q are the midpoint of the arcs AB and AC. Apply Pascal's theorem to $BACPTQ$ and we see that X, I, Y are collinear. Since I lies on the angle bisector of $\angle XAY$ and $AX = AY$, I must be the midpoint of XY. $\square$

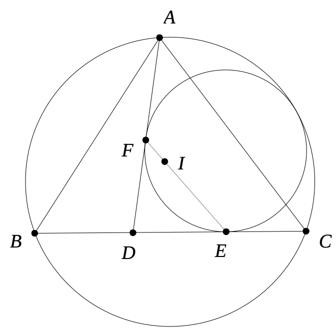
Related problems:

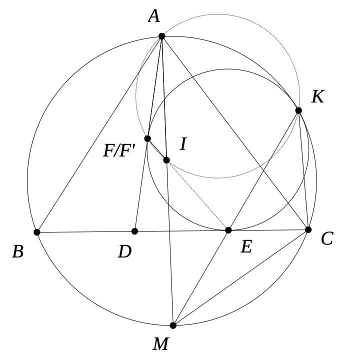
(i) (IMO 1978) In triangle ABC, $AB = AC$. A circle is tangent internally to the circumcircle of triangle ABC and also to sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC.

(ii) Let ABC be a triangle. Circle ω is tangent to AB and AC, and internally tangent to the circumcircle of triangle ABC. The circumcircle and ω are tangent at P. Let I be the incircle of triangle ABC. Line PI meets the circumcircle of ABC at P and Q. Prove that $BQ = CQ$.

6. **More curvilinear incircles.**

(A generalization of the previous lemma) Let ABC be a triangle, I its incenter and D a point on BC. Consider the circle that is tangent to the circumcircle of ABC but is also tangent to DC, DA at E, F respectively. Then E, F and I are collinear.





Proof. There is a "computational" proof using Casey's theorem[2] and transversal theorem[3]. You can try to work that out yourself. Here, we show a clever but difficult synthetic proof (communicated to me via Oleg Golberg).

Denote Ω the circumcircle of ABC and Γ the circle tangent tangent to the circumcircle of ABC and lines DC, DA. Let Ω and Γ touch at K. Let M be the midpoint of arc $\widehat{BC}$ on Ω not containing K. Then K, E, M are collinear (think: dilation with center K carrying Γ to Ω). Also, A, I, M are collinear, and $MI = MC$.

Let line EI meet Γ again at F'. It suffices to show that AF' is tangent to Γ.

Note that $\angle KF'E$ is subtended by $\widehat{KE}$ in Γ and $\angle KAM$ is subtended by $\widehat{KM}$ in Ω. Since $\widehat{KE}$ and $\widehat{KM}$ are homothetic with center K, we have $\angle KF'E = \angle KAM$, implying that A, K, I', F' are concyclic.

We have $\angle BCM = \angle CBM = \angle CKM$. So $\triangle MCE \sim \triangle MKC$. Hence $MC^2 = ME \cdot MK$. Since $MC = MI$, we have $MI^2 = ME \cdot MK$, implying that $\triangle MIE \sim \triangle MKI$. Therefore,

[2]**Casey's theorem**, also known as Generalized Ptolemy Theorem, states that if there are four circles $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ (could be degenerated into a point) all touching a circle Γ such that their tangency points follow that order around the circle, then

$$t_{12}t_{34} + t_{23}t_{14} = t_{13}t_{24},$$

where t_{12} is the length of the common tangent between Γ_i and Γ_j (if Γ_i and Γ_j on the same side of Γ, then take their common external tangent, else take their common internal tangent.) I think the converse is also true—if both equations hold, then there is some circle tangent to all four circles.

[3]The **transversal theorem** is a criterion for collinearity. It states that if A, B, C are three collinear points, and P is a point not on the line ABC, and A', B', C' are arbitrary points on lines PA, PB, PC respectively, then A', B', C' are collinear if and only if

$$BC \cdot \frac{AP}{A'P} + CA \cdot \frac{BP}{B'P} + AB \cdot \frac{CP}{C'P} = 0,$$

where the lengths are directed. In my opinion, it's much easier to remember the proof than to memorize this huge formula. The simplest derivation is based on relationships between the areas of $[PAB], [PA'B']$, etc.

$\angle KEI = \angle AIK = \angle AF'K$ (since A, K, I, F' are concyclic). Therefore, AF' is tangent to Ω and the proof is complete. $\square$

Related problems:

(i) (Bulgaria 2005) Consider two circles k_1, k_2 touching externally at point T. A line touches k_2 at point X and intersects k_1 at points A and B. Let S be the second intersection point of k_1 with the line XT . On the arc $\widehat{TS}$ not containing A and B is chosen a point C . Let CY be the tangent line to k_2 with $Y \in k_2$, such that the segment CY does not intersect the segment ST . If $I = XY \cap SC$. Prove that:

(a) the points C, T, Y, I are concyclic.

(b) I is the excenter of triangle ABC with respect to the side BC.

(ii) (Sawayama-Thébault[4]) Let ABC be a triangle with incenter I. Let D a point on side BC. Let P be the center of the circle that touches segments AD, DC, and the circumcircle of ABC, and let Q be the center of the circle that touches segments AD, BD, and the circumcircle of ABC. Show that P, Q, I are collinear.

(iii) Let P be a quadrilateral inscribed in a circle Ω, and let $\mathcal{Q}$ be the quadrilateral formed by the centers of the fourcircles internally touching O and each of the two diagonals of P. Show that the incenters of the four triangles having for sides the sides and diagonals of P form a rectangle R inscribed in Q.

(iv) (Romania 1997) Let ABC be a triangle with circumcircle Ω, and D a point on the side BC. Show that the circle tangent to Ω, AD and BD, and the circle tangent to Ω, AD and DC, are tangent to each other if and only if $\angle BAD = \angle CAD$.

(v) (Romania TST 2006) Let ABC be an acute triangle with $AB \neq AC$. Let D be the foot of the altitude from A and ω the circumcircle of the triangle. Let ω_1 be the circle tangent to AD, BD and ω. Let ω_2 be the circle tangent to AD, CD and ω. Let ℓ be the interior common tangent to both ω_1 and ω_2, different from CD. Prove that ℓ passes through the midpoint of BC if and only if $2BC = AB + AC$.

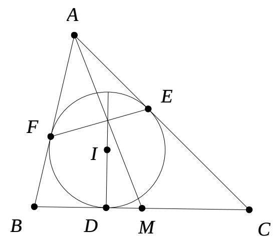
(vi) (AMM 10368) For each point O on diameter AB of a circle, perform the following construction. Let the perpendicular to AB at O meet the circle at point P. Inscribe circles in the figures bounded by the circle and the lines AB and OP. Let R and S be the points at which the two incircles to the curvilinear triangles AOP and BOP are tangent to the diameter AB. Show that $\angle RPS$ is independent of the position of O.

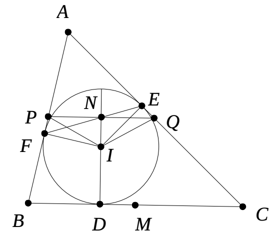
7. **Concurrent lines from the incircle.**

Let the incircle of ABC touch sides BC, CA, AB at D, E, F respectively. Let I be the incenter of ABC and M be the midpoint of BC. Then the lines EF, DI and AM are concurrent.

Proof. Let lines DI and EF meet at N. Construct a line through N parallel to BC, and let it meet sides AB and AC at P and Q, respectively. We need to show that A, N, M are collinear, so it suffices to show that N is the midpoint of PQ. We present two ways to finish this off, one using Simson's line, and the other using spiral similarities.

[4] A bit of history: this problem was posed by French geometer Victor Thébault (1882–1960) in the *American Mathematical Monthly* in 1938 (Problem 2887, 45 (1938) 482–483) and it remained unsolved until 1973. However, in 2003, Jean-Louis Ayme discovered that this problem was independently proposed and solved by instructor Y. Sawayama of the Central Military School of Tokyo in 1905! For more discussion, see Ayme's paper at `http://forumgeom.fau.edu/FG2003volume3/FG200325.pdf`





Simson line method: Consider the triangle APQ. The projections of the point I onto the three sides of APQ are D, N, F, which are collinear, I must lie on the circumcircle of APQ by Simson's theorem. But since AI is an angle bisector, $PI = QI$, thus $PN = QN$.

Spiral similarity method: Note that P, N, I, F are concyclic, so $\angle EFI = \angle QPI$. Similarly, $\angle PQI = \angle FEI$. So triangles PIQ and FIE are similar. Since $FI = EI$, we have $PI = QI$, and thus $PN = QN$. (c.f. Lemma 3) □

Related problems:

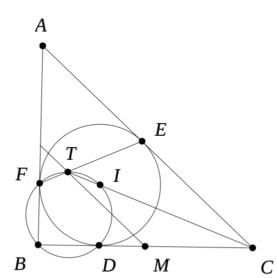
(i) (China 1999) In triangle ABC, $AB \neq AC$. Let D be the midpoint of side BC, and let E be a point on median AD. Let F be the foot of perpendicular from E to side BC, and let P be a point on segment EF. Let M and N be the feet of perpendiculars from P to sides AB and AC, respectively. Prove that M, E, and N are collinear if and only if $\angle BAP = \angle PAC$.

(ii) (IMO Shortlist 2005) The median AM of a triangle ABC intersects its incircle ω at K and L. The lines through K and L parallel to BC intersect ω again at X and Y. The lines AX and AY intersect BC at P and Q. Prove that $BP = CQ$.

8. **More circles around the incircle.**

Let I be the incenter of triangle ABC, and let its incircle touch sides BC, AC, AB at D, E and F, respectively. Let line CI meet EF at T. Then T, I, D, B, F are concyclic. Consequent results include: $\angle BTC = 90°$, and T lies on the line connecting the midpoints of AB and BC.

An easier way to remember the third part of the lemma is: for a triangle ABC, draw a midline, an angle bisector, and a touch-chord, each generated from different vertex, then the three lines are concurrent.



Proof. Showing that I, T, E, B are concyclic is simply angle chasing (e.g. show that $\angle BIC = \angle BFE$). The second part follows from $\angle BTC = \angle BTI = \angle BFI = 90°$. For the third part, note that if M is the midpoint of BC, then M is the midpoint of an hypotenuse of the right triangle BTC. So $MT = MC$. Then $\angle MTC = \angle MCT = \angle ACT$, so MT is parallel to AC, and so MT is a midline of the triangle. □

Related problems:

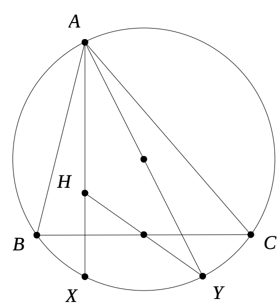
(i) Let ABC be an acute triangle whose incircle touches sides AC and AB at E and F, respectively. Let the angle bisectors of $\angle ABC$ and $\angle ACB$ meet EF at X and Y, respectively, and let the midpoint of BC be Z. Show that XYZ is equilateral if and only if $\angle A = 60°$.

(ii) (IMO Shortlist 2004) For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

(iii) Let points A and B lie on the circle Γ, and let C be a point inside the circle. Suppose that ω is a circle tangent to segments AC, BC and Γ. Let ω touch AC and Γ at P and Q. Show that the circumcircle of APQ passes through the incenter of ABC.

9. **Reflections of the orthocenter lie on the circumcircle.**

Let H be the orthocenter of triangle ABC. Let the reflection of H across the BC be X and the reflection of H across the midpoint of BC be Y. Then X and Y both lie on the circumcircle of ABC. Moreover, AY is a diameter of the circumcircle.



Proof. Trivial. Angle chasing. □

Related problems:

(i) Prove the existence of the nine-point circle. (Given a triangle, the nine-point circle is the circle that passes through the three midpoints of sides, the three feet of altitudes, and the three midpoints between the orthocenter and the vertices).

(ii) Let ABC be a triangle, and P a point on its circumcircle. Show that the reflections of P across the three sides of ABC lie on a lie that passes through the orthocenter of ABC.

(iii) (IMO Shortlist 2005) Let ABC be an acute-angled triangle with $AB \neq AC$, let H be its orthocentre and M the midpoint of BC. Points D on AB and E on AC are such that $AE = AD$ and D, H, E are collinear. Prove that HM is orthogonal to the common chord of the circumcircles of triangles ABC and ADE.

(iv) (USA TST 2005) Let $A_1A_2A_3$ be an acute triangle, and let O and H be its circumcenter and orthocenter, respectively. For $1 \le i \le 3$, points P_i and Q_i lie on lines OA_i and $A_{i+1}A_{i+2}$ (where $A_{i+3} = A_i$), respectively, such that OP_iHQ_i is a parallelogram. Prove that

$$\frac{OQ_1}{OP_1} + \frac{OQ_2}{OP_2} + \frac{OQ_3}{OP_3} \ge 3.$$

(v) (China TST quizzes 2006) Let ω be the circumcircle of triangle ABC, and let P be a point inside the triangle. Rays AP, BP, CP meet ω at A_1, B_1, C_1, respectively. Let A_2, B_2, C_2 be the images of A_1, B_1, C_1 under reflection about the midpoints of BC, CA, AB, respectively. Show that the orthocenter of ABC lies on the circumcircle of $A_2B_2C_2$.

10. ***O* and *H* are isogonal conjugates.**

Let ABC be a triangle, with circumcenter O, orthocenter H, and incenter I. Then AI is the angle bisector of $\angle HAO$.

Proof. Trivial. □

Related problems:

(i) (Crux) Points O and H are the circumcenter and orthocenter of acute triangle ABC, respectively. The perpendicular bisector of segment AH meets sides AB and AC at D and E, respectively. Prove that $\angle DOA = \angle EOA$.

(ii) Show that $IH = IO$ if and only if one of $\angle A$, $\angle B$, $\angle C$ is $60°$.